Thomas J. Watson Research Center, Yorktown Heights

ON THE LINE GRAPH OF A PROJECTIVE PLANE+

by

A. J. Hoffman

9/18/63

1. Introduction

If G is a (finite, undirected) graph, its line graph (also called the interchange graph, and the adjoint graph) is the graph G^* whose vertices are the edges of G, with two vertices of G^* adjacent if the corresponding edges of G are adjacent. Let π be a projective plane with n + 1 points on a line, and let $G(\pi)$ be the bipartite graph whose vertices are the $2(n^2 + n + 1)$ points and lines of π, with two vertices adjacent if and only if one of the vertices is a point, the other is a line, and the point is on the line. The graph ~~we shall study is~~ $(G(\pi))^*$

For any graph G, let

$$A(G) = A = (a_{ij}) = \left\{\begin{matrix} 1 \text{ if } i \text{ and } j \text{ are adjacent vertices} \\ 0 \text{ otherwise} \end{matrix}\right\}.$$

A is called the adjacency matrix of G, and in recent years there have been several investigations to determine to what extent a regular, connected graph is determined by the characteristic roots of its adjacency matrix. In the case where G is a line graph, the following

+This research was supported in part by the Office of Naval Research under Contract No. Nonr 3775(00), NR 047040.

results have been obtained:

(i) if G is the line graph of the complete bipartite graph on $n + n$ vertices, and H is a regular connected graph on n^2 vertices such that $A(H)$ has the same characteristic roots as $A(G)$, then $H \cong G$ unless $n = 4$, when there is exactly one exception [9].

(ii) if G is the line graph of the complete graph on n vertices, and H is a regular connected graph on $n(n - 1)/2$ vertices, such that $A(H)$ has the same characteristic roots as $A(G)$, then $H \cong G$, unless $n = 8$, when there are exactly two exceptions ([1], [2], [3], [4], [5], [8]).

In this paper, we shall prove that if H is a regular connected graph on $(n + 1)(n^2 + n + 1)$ vertices such that $A(H)$ has the same characteristic roots as $A((G(\pi))^*)$, then $H \cong (G(\pi_1))^*$, where π_1 is some projective plane of the same order as π. Thus the characteristic roots of $A((G(\pi))^*)$ do determine the class of graphs $(G(\pi))^*$, but do not distinguish between projective planes of the same order.

2. The Characteristic Roots of $A((G(\pi))^*)$.

It is useful to first determine the characteristic roots of $A(G(\pi))$.

Lemma 1: A regular connected graph G on $2(n^2 + n + 1)$ vertices has as the distinct characteristic roots of $A(G)$

(2.1) $$(n + 1),\ -(n + 1),\ \sqrt{n},\ -\sqrt{n}$$

if and only if $G = G(\pi)$, where π is a projective plane of the order n.

Proof: By definition, if $G = G(\pi)$,

$$A(G) = \begin{pmatrix} O & B \\ B^T & O \end{pmatrix}, \tag{2.2}$$

where B is a point-line incidence matrix of π. The characteristic roots of (2.2) are the singluar values of B and their negatives. But the singular values of B are $n + 1$ and $\sqrt{n}$ ([7]).

Conversely, assume A has (2.1) as its distinct characteristic roots. Then ([6]), if $A = A(G)$, because G is bipartite, A is of the form (2.2), where B is a (0, 1) matrix with row and column sums equal to $n + 1$, and BB^T has all but one characteristic root equal to n. Hence $BB^T - nI$ is a nonnegative integral symmetric matrix of rank one with every diagonal entry equal to 1. This implies $BB^T - nI = J$; i.e., B is the incidence matrix of a projective plane π of order n.

Another derivation of lemma 1 is given in the thesis of R. R. Singleton [10], in which it is proved that a regular connected graph H of valence $n + 1$ and girth 6 has $2(n^2 + n + 1)$ vertices if and only if $H = G(\pi)$.

<u>Lemma 2:</u> The distinct characteristic roots of $A(G(\pi)^*)$ are

$$2n, \; -2, \; n - 1 \pm \sqrt{n} . \tag{2.3}$$

Proof: Let $A = A((G(\pi))^*)$, B be the adjacency matrix for $G(\pi)$. Let K be the $2(n^2 + n + 1)$ by $(n + 1)(n^2 + n + 1)$ matrix

whose rows correspond to the points and lines of π, and whose columns correspond to the edges of $(G(\pi))^*$; i.e., each colum of K contains two 1's, corresponding to an incident point and line of π, the remaining entry in the column being 0. Clearly,

$$KK^T = (n + 1) I + B, \quad K^TK = 2I + A \ .$$

The distinct characteristic roots of KK^T and K^TK are the same except possibly for 0. But K^TK is singular, since its rank is at most $2(n^2 + n + 1)$, while its order is $(n + 1)(n^2 + n + 1)$; KK^T is singular, since the sum of the rows of K corresponding to points of π minus the sum of the rows of K corresponding to lines of π is the zero vector. Thus the distinct eigenvalues of KK^T and of K^TK are the same. Invoking (2.1) then proves (2.3).

3. <u>Theorem</u>

If G is a regular connected graph with no edges joining a vertex to itself, if G has $(n + 1)(n^2 + n + 1)$ vertices and the adjacency matrix of G has (2.3) as its distinct eigenvalues, then $G = (G(\pi))^*$, for some projective plane π of order n.

In the lemmas that follow, we assume that G satisfies the hypothesis of the theorem, $A = A(G)$, J is the matrix every entry of which is 1.

<u>Lemma 3:</u> Let

(3.1) $\quad P(x) = \frac{1}{2}\left(x^3 - (2n - 4)x^2 + (n^2 - 7n + 5)x + 2(n^2 - 3n + 1)\right).$

then $P(A) = J$.

Proof: It has been shown [6] that the adjacency matrix of a regular connected graph of valence d on N vertices, with distinct eigenvalues $d, \alpha_1, \ldots \alpha_t$, satisfies $P(B) = J$, where $P(x) = N \prod_i (x - \alpha_i)/\prod_i (d - \alpha_1)$. From (2.3), we then calculate (3.1).

<u>Lemma 4:</u> If two vertices of G are adjacent, then there are exactly $n - 1$ vertices of G adjacent to both. If two vertices of G are not adjacent, then there are no vertices or exactly one vertex adjacent to both.

Proof: Let i be any vertex of G. Then i has valence $2n$, so there are $2n$ vertices $j_1, \ldots, j_{2n}$ such that $a_{ij_t} = 1$, $t = 1, \ldots, 2n$. We first show that

$$\sum_t (A^2)_{ij_t} = 2n(n-1) . \tag{3.2}$$

This follows from (3.1); for the left side of (3.2) is $(A^3)_{ii}$, and by (3.1), $(A^3)_{ii} = 2(J)_{ii} + (2n - 4)(A^2)_{ii} - (n^2 - 7n + 5) A_{ii} - 2(n^2 - 3n + 1)$. But $J_{ii} = 1$, $(A^2)_{ii} = 2n$, $A_{ii} = 0$, and (3.2) follows.

Next, consider the matrix

$$B = A^2 - 2nI - (n-1) A \tag{3.3}$$

We shall show that every entry of B is 0 or 1. Certainly every entry is an integer. Let i be any row of B. From the fact that $\sum_j (A^2)_{ij} = (2n)^2$, we infer that

$$\sum_j b_{ij} = 2n^2 . \tag{3.4}$$

We next evaluate $\sum_j b_{ij}^2 = (B^2)_{ii}$. We have from (3.3)

(3.5) $$B^2 = A^4 - 2(n-1)\,A^3 + (n^2 - 6n + 1)\,A^2 + 4n\,(n-1)\,A + 4n^2\,I\,.$$

Further, $I_{ii} = 1$, $A_{ii} = 0$, $A^2_{ii} = 2n$, $A^3_{ii} = 2n\,(n-1)$ from (3.2). To evaluate $(A^4)_{ii}$, we use (3.1), with $P(A) = J$, and obtain $A\,P(A) = AJ = 2n\,J$. Since $A\,P(A)$ is a fourth degree polynomial in A, we can evaluate

$$(A^4)_{ii} = 4n - 2n\,(n^2 - 7n + 5) + 2n\,(n-1)\,(2n-4)\,.$$

Putting these expressions in (3.5), we obtain

(3.6) $$(B^2)_{ii} = \sum_j b_{ij}^2 = 2n^2\,.$$

From (3.5) and (3.6) we infer that each of the integers b_{ij} is 0 or 1. Recalling the definition of B in (3.3), this proves the second sentence of the lemma. To prove the first sentence, note from (3.2) and (3.3) that $\sum_t b_{ij_t} = 0$. Since each b_{ij} is 0 or 1, each $b_{ij_t} = 0$. By (3.3), this proves the first sentence of the lemma.

Lemma 5: G contains $2(n^2 + n + 1)$ cliques $C_1, \ldots, C_{2(n^2+n+1)}$ with the following properties:

(3.7) Each C_i contains exactly $n + 1$ vertices

(3.8) Each vertex of G is contained in exactly two C_i

(3.9) Each pair of adjacent vertices of G is contained in exactly one C_i.

Proof: The set of cliques C_i will consist of all cliques with $n + 1$ vertices, which establishes (3.7). To prove (3.9), let i and

j be adjacent vertices of G. Let k and ℓ each be adjacent to both i and j. If k and ℓ were not adjacent, we would have a violation of the second sentence of lemma 4. Hence, the $n - 1$ vertices adjacent to both i and j (by the first sentence of lemma 4) are adjacent to each other. These vertices, together with i and j are the unique clique with $n + 1$ vertices containing i and j.

Let T be the total number of $(n + 1)$ - cliques, and let us count the number of incidences of cliques with pairs of vertices contained in the clique. This is

$$T \binom{n+1}{2} = \frac{1}{2}\, 2n\, (n + 1)\, (n^2 + n + 1),$$

for the right hand side is the total number of pairs of adjacent vertices. This equation yields $T = 2\,(n^2 + n + 1)$. Thus all that remains to be proven is (3.8). Since the valence of each vertex i is $2n$, there must be at least two $(n + 1)$ - cliques containing i. If these two cliques did not contain all vertices adjacent to i, there would have to be some vertex $j \neq i$ in both cliques, violating (3.9).

We are now ready to prove the theorem. Let $\tilde{G}$ be the graph whose vertices are the $(n + 1)$ - cliques of G. Two vertices of $\tilde{G}$ are adjacent if the corresponding cliques of G have a common vertex. It follows from lemma 5 that $\tilde{G}$ is a regular connected graph of valence $n + 1$, and that $G = \tilde{G}^*$. We will be finished if we

prove that $\tilde{G} = G(\pi)$. Let L be the vertex-edge incidence matrix of $\tilde{G}$, and let $\tilde{A}$ be the adjacency matrix of $\tilde{G}$. Assume $\tilde{A}$ has distinct characteristic roots $n + 1$, $\alpha_1, \ldots, \alpha_t$. Since

$$LL^T = (n+1)I + \tilde{A}, \quad L^T L = 2I + A,$$

and (except possibly for 0) the distinct characteristic roots of LL^T and $L^T L$ are the same, it follows by the same reasoning as in Lemma 2 that the distinct characteristic roots (with the possible exception of -2) of A are

$$2n, \quad n - 1 + \alpha_t. \tag{3.10}$$

Comparing (3.10) with (2.3), we see that, if -2 is of the form $n - 1 + \alpha_\epsilon$, then $\tilde{A}$ has the same distinct characteristic roots as the adjacency matrix for $G(\pi)$, and (by the "only if" part of lemma 1) we are finished. Therefore, assume otherwise, so that (comparing (3.10) with (2.3)) we find that the distinct characteristic roots of $\tilde{A}$ are

$$n + 1, \ \pm\sqrt{n} .$$

Since $\tilde{G}$ is regular and connected, we can, as in lemma 3, use the theorem of [6] to assert that

$$2(\tilde{A}^2 - nI) = J .$$

But since $\tilde{A}$ is a (0, 1) matrix, this is absurd.

REFERENCES

[1] L. C. Chang, The Uniqueness and Nonuniqueness of the Triangular Association Scheme, Science Record 3 (1959) 604-613.

[2] ——————, Association Schemes of Partially Balanced Block Designs with Parameters $\nu = 28$, $n_1 = 12$, $n_2 = 15$ and $p_{11}^2 = 4$", Science Record 4 (1960) 12-18.

[3] W. S. Connor, The Uniqueness of the Triangular Association Scheme, Annals of Mathematical Statistics 29, (1958) 262-266.

[4] A. J. Hoffman, On the Uniqueness of the Triangular Association Scheme, Annals of Mathematical Statistics 31, (1960) 492-497.

[5] ——————, On the Exceptional Case in a Characterization of the Arcs of a Complete Graph, IBM Journal of Research Development 4 (1961) 487-496.

[6] ——————, On the Polynomial of a Graph, American Mathematical Monthly 70 (1963) 30-36.

[7] H. J. Ryser, Geometries and Incidence Matrices, American Mathematical Monthly 62 (1955), No. 7, part II, 25-31.

[8] S. S. Shrikhande, On a Characterization of the Triangular Association Scheme, Annals of Mathematical Statistics 30 (1959) 39-47.

[9] ——————, The Uniqueness of the L_2 Association Scheme, ibid. 30 (1959) 781-798.

[10] R. R. Singleton, On Minimal Graphs of Even Girth, Princeton thesis, 1962.

LIMITED DISTRIBUTION NOTICE

This report has been submitted for publication elsewhere and has been issued as a Research Note for early dissemination of its contents. As a courtesy to the intended publisher, it should not be widely distributed until after the date of outside publication.

Printed in U.S.A.